Finite-size scaling for directed bond percolation with and without cycles on a triangular lattice

1986 J. Phys. A: Math. Gen. 19 L527

(http://iopscience.iop.org/0305-4470/19/9/012)

J. Phys. A: Math. Gen. 19 (1986) L527-L532. Printed in Great Britain

LETTER TO THE EDITOR

Finite-size scaling for directed bond percolation with and without cycles on a triangular lattice†

Anna Chame‡, S L A de Queiroz and R R dos Santos
Departamento de Física, PUC, 22452 Rio de Janeiro, Brazil

Received 3 March 1986

Abstract. We apply finite-size scaling and phenomenological renormalisation group arguments to the problems of directed acyclic, directed cyclic and undirected bond percolation on a triangular lattice. Our results are in good agreement with known estimates, and show that the phenomenological renormalisation procedure is sensitive to the difference between global and local directional biases, as well as to the distinction between locally directed and fully isotropic problems. In addition, we draw tentative comments on the extension, to directed problems, of the relation $A = \pi\eta$ (where A is the inverse correlation length amplitude of a strip of finite width at criticality and η is the exponent that describes the decay of correlations at the critical point), known to hold for various isotropic systems.

Finite-size scaling theory (Fisher 1971; for a recent review see Barber 1983) has been most successful in deriving critical properties of infinite systems from those of their finite counterparts. In particular, the conjugation of transfer matrix methods with finite-size scaling in the phenomenological renormalisation group (Nightingale 1976) has been a powerful tool in obtaining accurate values of critical parameters (temperatures and exponents) for a number of physical models (Nightingale 1982, Barber 1983). Although convergence is eventually reached in most cases, it often happens that phenomenological renormalisation estimates approach their limiting values faster for thermal problems than for geometric ones. This can be seen, e.g., in two dimensions if, for strips with a given small width $\leqslant 5$ sites, one compares results obtained for Ising (Nightingale 1976) or transverse Ising models (e.g. dos Santos and Sneddon 1981) and the corresponding ones, e.g., for percolation and lattice animals (Derrida and Vannimenus 1980, Derrida and De Seze 1982) or self-avoiding walks (Derrida 1981). For wider strips errors are usually small and convergence monotonic in either type of problem.

The existence of a directional bias has been successfully accounted for in phenomenological renormalisation schemes; see Domany and Kinzel (1981) and Kinzel and Yeomans (1981) for directed percolation and Nadal *et al* (1982) for directed animals.

In the present letter we discuss the application of phenomenological renormalisation to the problem of directed bond percolation on a triangular lattice; for a review of properties and applications of directed percolation, see Kinzel (1983). The distinguishing feature of this case, as compared, e.g., to directed percolation on a square lattice,

† Work supported by Brazilian agencies FINEP, CNPq and CAPES.
‡ Present address: Centro Brasileiro de Pesquisas Físicas, Rua Xavier Sigaud 150, 22290 Rio de Janeiro, Brazil.

is that the directional bias may be global or only of a local nature; this depends exclusively on the relative disposition of bond directionalities along the three lattice directions. As can be seen from figure 1, if bond directionalities are distributed as in figure 1(*a*) (acyclically) the overall preferred direction of, e.g., information flow is from left to right, whereas with the arrangement of figure 1(*b*) (cyclical), directional constraints only have a local effect and the system is globally isotropic. Accordingly, series studies (Blease 1977a,b,c) indicate that percolation on a triangular lattice with cycles belongs to the same universality class as undirected percolation, while the acyclical case clearly does not. Our first purpose here is to check the extent to which the phenomenological renormalisation method is sensitive to the difference between so-called 'global' and 'local' directional biases. In doing so we expect to contribute towards a clearer understanding of the advantages and limitations (if any) of the method. Secondly, we make brief comments on the possible extension to directed problems of a universality property of critical amplitudes in finite-size scaling, known to hold for isotropic systems (see e.g. Privman and Fisher 1984 and references therein).

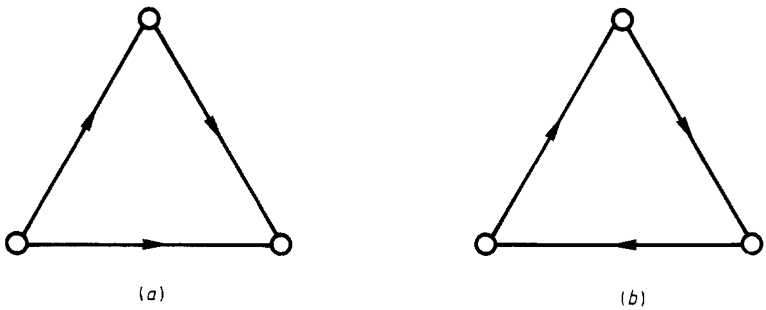

Figure 1. Directed bonds on a triangular lattice, arranged so that in (*a*) there is a preferred direction (left to right) whereas in (*b*) there is no overall favoured direction.

Apart from the series work of Blease (1977a,b,c) quoted above, little has been done on directed percolation on a triangular lattice. Examples are the exact solution of Wu and Stanley (1982) of a particular case where all bonds are present along a given direction, and the phenomenological renormalisation study of partially directed site percolation of Martin and Vannimenus (1985); the latter authors use their results jointly with those for the same problem on a square lattice, with the main purpose of discussing corrections to scaling. Perhaps the lack of interest is because no qualitative difference is expected from the same problem on a square lattice (which has been much more extensively studied), with the exception of the case discussed here, where formation of vortices (cycles) destroys global directionality.

Throughout our work we have used periodic boundary conditions, with which one usually obtains sequences of estimates that extrapolate more smoothly than if free boundary conditions are imposed (Derrida and Vannimenus 1980, Derrida 1981). For the acyclic problem we have used strips of width $N = 2, 3, 4$ and 5 (see figure 2), with the preferred direction along the strip. As is known from anisotropic scaling (Kinzel and Yeomans 1981, Kinzel 1983), since there are three critical parameters (the critical probability p_c and the critical exponents $\nu_\parallel$ and $\nu_\perp$ defined by $\xi_\parallel \sim (p_c - p)^{-\nu_\parallel}$ and $\xi_\perp \sim (p_c - p)^{-\nu_\perp}$, where $\xi_\parallel$ and $\xi_\perp$ are correlation lengths respectively along and perpendicular to the preferred direction). The most efficient way is to compare three strips of successive widths in order to obtain estimates for the unknown quantities contained

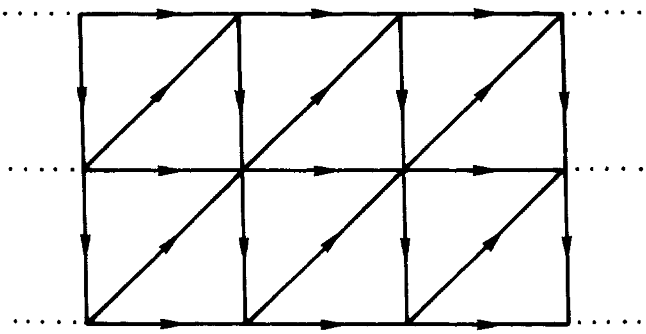

Figure 2. Strip of width $N = 3$ sites used in our calculations. Here, bonds are directed acyclically. Periodic boundary conditions are used; for clarity, the corresponding bonds are not represented.

in the recursion relation; p^* (the critical probability) and θ (the anisotropy exponent $\equiv \nu_{\parallel}/\nu_{\perp}$). The exponent $\nu_{\perp}$ can then be obtained from the appropriate derivatives of the perpendicular correlation length $\xi_{\perp}$ (which scales linearly with N) at the fixed point (Kinzel and Yeomans 1981). Since it is $\xi_{\parallel}$ that is given by the largest eigenvalue of the transfer matrix, we use the relation $\xi_{\perp} = \xi_{\parallel}^{1/\theta}$. For the cyclic case, we have used only strips of widths 2 and 3; this is because the number of distinct configurations which have to be counted for the transfer matrix grows much faster than even in undirected percolation (see Derrida and Vannimenus (1980) for an example of configurations in undirected percolation). The problem here is that one has to distinguish whether a site in column N is connected to column 1 (Derrida and Vannimenus 1980) by a path of bonds directed either *from* column N *to* column 1 or vice versa. Already at width $N = 3$ the transfer matrix is 22×22, which makes it quite impractical to push the calculations further towards larger N. In analysing data for the cyclic problem, we have assumed the existence of a unique, direction-independent, correlation length exponent; thus, estimates for p_c and ν can be obtained from comparison of a pair of strips in the usual way (see e.g. Derrida and Vannimenus 1980). In order to check whether this assumption had too strong a bias towards identifying the cyclic problem with the fully undirected one, we have calculated p_c and ν for undirected percolation on a triangular lattice, using strips of width 2 and 3. Our results are displayed in table 1, for all cases studied. The following comments are in order.

(i) For the acyclic case, estimates for the critical probability are in better agreement with known results than those for exponents; however, the latter do show the correct trend with increasing N. A naive two-point extrapolation against $1/N^2$ (this variable seems to be the best choice for systems with periodic boundary conditions (Barber and Fisher 1973)) gives $\theta_{\text{ext}} \simeq 1.34$ and $\nu_{\perp\text{ext}} \simeq 1.13$. For the site problem on a square lattice, Kinzel and Yeomans (1981) quote $\theta \simeq 1.54$ and $\nu_{\perp} \simeq 1.13$ before extrapolation, from strips with $N = 3$, 4 and 5; this shows that our problem behaves slightly worse than usual, as regards the rate of convergence with increasing N (see the remarks in the introductory paragraph).

(ii) Comparison between results for cyclic and undirected cases shows that critical probability estimates differ widely (while, for each given case, our estimate is in very good agreement with data from other sources); on the other hand, critical exponents are rather close to each other and to the accepted value for undirected percolation. The picture is doubtless consistent with the same universal behaviour for cyclic and

Table 1. Critical parameters calculated by the phenomenological renormalisation group for directed acyclic, directed cyclic and undirected bond percolation on a triangular lattice.

Acyclic	$N=2,3,4$		$N=3,4,5$	Other estimates
p_c	0.4608		0.4657	0.479 ± 0.003[a]
θ	1.2241		1.2724	1.581 ± 0.001[b]
$\nu_\perp$	1.2146		1.1773	1.094 ± 0.001[b]
Cyclic		$N=2,3$		Other estimates
p_c		0.5660		0.571 ± 0.003[c]
ν		1.2678		1.25 ± 0.05[c]
Undirected		$N=2,3$		Other estimates
p_c		0.3471		$2\sin\frac{1}{18}\pi=0.3473\ldots$[d]
ν		1.2644		$\frac{4}{3}$[e]

[a] Blease (1977c).
[b] Kinzel and Yeomans (1981) (square lattice).
[c] Blease (1977a).
[d] Sykes and Essam (1963).
[e] den Nijs (1979).

undirected percolation. On the other hand, from the above extrapolated values of θ and $\nu_\perp$ for the acyclic case, one obtains $\nu_\parallel(=\theta\nu_\perp)\simeq1.51$ which is clearly distinct from the estimates for both acyclic and undirected percolation. Note that the comparison must be between ν for the isotropic problem and $\nu_\parallel$ for the directed one, because in the latter case it is $\nu_\parallel$ which relates to an actual correlation length (Kinzel 1983).

(iii) Although, for the small strip widths used here, we cannot claim the usual accuracy of ~1 part in 10^4 exhibited by phenomenological renormalisation groups with $N\geqslant10$, the overall picture shows that the phenomenological renormalisation group is properly sensitive to the difference between global and local bias, and also to the small nuance between local bias and full isotropy.

Still with regard to the application of finite-size scaling concepts to the problem of directed percolation, we wish to comment briefly on the feasibility of an extension, to directed problems, of a recently found relationship between critical exponents and finite-size scaling amplitudes. For an undirected system on a two-dimensional strip of linear width L, if all parameters such as temperature and symmetry-breaking field are set to the critical values of the truly infinite system, the inverse correlation length scales with L as $\xi^{-1}=A/L$. The amplitude A has been shown to be related to the exponent η, which describes the decay of spin-spin correlation functions at criticality, by

$$A=\pi\eta. \tag{1}$$

This is true, e.g., for the case of Anderson localisation (Pichard and Sarma 1981), the XY model (Luck 1982), the q-state Potts model with $q<4$ (thus including the Ising model ($q=2$) and bond percolation ($q\to1$)) (Derrida and De Seze 1982) and the Ashkin-Teller model (Alcaraz and Drugowich de Felicio 1984); in particular, Nightingale and Blöte (1983) worked out an extension of (1) for anisotropic (though undirected) systems. Privman and Fisher (1984) discussed the general grounds for universality of scaling functions; Cardy (1984) showed how (1) above could be obtained from conformal invariance.

Although the fact that conformal invariance does not hold for systems with a directional bias may be a deterrent to attempts of generalisation of (1) to directed problems, we decided to check our numerical data and see whether a promising picture emerged. Since in finite-size scaling for directed systems it is the perpendicular correlation length which scales linearly with N, we plotted $\xi_\perp/N$ against $1/N$ for the acyclic problem (following, e.g., Burkhardt and Guim 1985), with $\xi_\perp$ calculated at the best available estimate of p_c (actually as p_c is not known exactly (see table 1) we made plots for different values of p_c within the error bars). Since it is $\xi_\|$ which is obtained from the eigenvalue of the transfer matrix, we made use of $\xi_\perp = \xi_\|^{1/\theta}$, with $\theta = 1.581$ (the central estimate of Kinzel and Yeomans (1981)). Taking into account the fact that it is the actual strip width which enters into the finite-size relation $\xi^{-1} = A/L$ (Privman and Fisher 1984) we see that for the triangular lattice an additional factor of $\sqrt{3}/2$ enters (see figure 2) when one tries to extract A from an extrapolation of ξ/N against $1/N$ (N = number of sites). With $N = 2$–5, the straightest plot (to 3 parts in 10^4) for different assumed p_c within the error bars was, not surprisingly, for $p_c = 0.479$, the central estimate of Blease (1977c). From this plot, we obtain an extrapolated value of $A \simeq 0.594$ (for p_c in the range 0.479 ± 0.002, A (extrapolated) suffers a fluctuation of ± 0.020). If now we recall that in Cardy's argument the factor of π originated from the transformation of the whole plane (because isotropic correlations spread over it all) onto a strip of width 2π, we see that since in the present case correlations spread only forward owing to directionality, it is only a half-plane which should be mapped onto a strip (thus of width π) in the corresponding transformation for the directed problem. If this is true, we must have

$$A = \tfrac{1}{2}\pi\eta_\| \tag{2}$$

in the case where $\eta_\|$ is present because it is the decay of correlations along the strip which matters (Nightingale and Blöte 1983), and in our case the strip is parallel to the preferred direction.

Although the above arguments are certainly open to challenge, it is interesting to compare the value of $\eta_\| \simeq 0.378$ obtained from (2) to that obtained from $\eta_\| = 2\beta/\nu_\|$, with $\beta = 0.28 \pm 0.01$ and $\nu_\| = 1.732 \pm 0.001$ (Kinzel 1983), namely $\eta_\| = 0.32 \pm 0.01$. The error (15%) is the same as if one compares our extrapolated $\theta(\sim 1.34)$ with the best estimate for that exponent (1.581). Whether this is purely a numerical coincidence or has some deeper meaning, we do not know at present. In our opinion, it would be interesting if the existence of a relation between critical exponents and finite-size scaling amplitudes for directed problems could be definitely proved (or else disproved by positive arguments). An analysis of the problem on the square lattice, where a wealth of data are available, would be welcome.

In summary, we have discussed the application of finite-size scaling and phenomenological renormalisation group arguments to the problems of directed acyclic, directed cyclic and undirected bond percolation on a triangular lattice. We have found that our calculational method is sensitive to the differences between the various problems, thus showing its reliability under conditions hitherto untested, to our knowledge. In addition, we have made a few tentative remarks on the feasibility of an extension, to directed problems, of a universality property of critical amplitudes in finite-size scaling which is known to hold for undirected systems. Although our results on this latter point are inconclusive, we hope our preliminary approach will motivate researchers to further work on this interesting problem.

We wish to thank F C Alcaraz, J R Drugowich de Felicio, H J Herrmann, R Koberle, P M Oliveira and C Tsallis for helpful discussions and interesting suggestions.

References

Alcaraz F C and Drugowich de Felicio J R 1984 *J. Phys. A: Math. Gen.* **17** L651
Barber M N 1983 *Phase Transitions and Critical Phenomena* vol 7, ed C Domb and J L Lebowitz (New York: Academic) p 145
Barber M N and Fisher M E 1973 *Ann. Phys., NY* **77** 1
Blease J 1977a *J. Phys. C: Solid State Phys.* **10** 917
—— 1977b *J. Phys. C: Solid State Phys.* **10** 925
—— 1977c *J. Phys. C: Solid State Phys.* **10** 3461
Burkhardt T W and Guim I 1985 *J. Phys. A: Math. Gen.* **18** L25
Cardy J L 1984 *J. Phys. A: Math. Gen.* **17** L385
den Nijs M P M 1979 *J. Phys. A: Math. Gen.* **12** 1857
Derrida B 1981 *J. Phys. A: Math. Gen.* **14** L5
Derrida B and De Seze L 1982 *J. Physique* **43** 475
Derrida B and Vannimenus J 1980 *J. Physique Lett.* **41** 473
Domany E and Kinzel W 1981 *Phys. Rev. Lett.* **47** 5
dos Santos R R and Sneddon L 1981 *Phys. Rev.* B **23** 3541
Fisher M E 1971 *Critical Phenomena: Enrico Fermi Summer School Varenna, Italy* course 51, ed M S Green (New York: Academic) p 1
Kinzel W 1983 *Percolation Structures and Processes* ed G Deutscher, R Zallen and J Adler (Bristol: Adam Hilger) p 425
Kinzel W and Yeomans J M 1981 *J. Phys. A: Math. Gen.* **14** L163
Luck J M 1982 *J. Phys. A: Math. Gen.* **15** L169
Martin H O and Vannimenus J 1985 *J. Phys. A: Math. Gen.* **18** 1475
Nadal J P, Derrida B and Vannimenus J 1982 *J. Physique* **42** 1561
Nightingale M P 1976 *Physica* **83A** 561
—— 1982 *J. Appl. Phys.* **53** 7927
Nightingale P and Blöte H 1983 *J. Phys. A: Math. Gen.* **16** L657
Pichard J L and Sarma G 1981 *J. Phys. C: Solid State Phys.* **14** L617
Privman V and Fisher M E 1984 *Phys. Rev.* B **30** 322
Sykes M F and Essam J W 1963 *Phys. Rev. Lett.* **10** 3
Wu F Y and Stanley H E 1982 *Phys. Rev. Lett.* **48** 775